Performing an analysis similar to that in /8/, we obtain that the function (2.2) in this example has the form

$$\beta(t, z) = \frac{e^{at}-1}{a}\rho - \frac{e^{bt}-1}{b}\sigma, \quad t \geqslant 0$$

Then, if $a<0, b<0, \rho \geqslant \sigma$, and $\rho/a < \sigma/b$, Theorem 2 is applicable, and it guarantees complete controllability of process (4.4).

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Translated by Z.L.

PMM U.S.S.R., Vol.54, No.6, pp. 744-751, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

THE EXISTENCE AND STABILITY OF INVARIANT SETS OF DYNAMICAL SYSTEMS*

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The possibility of using Lyapunov functions to construct invariant sets of dynamical systems is discussed. The investigations presented herein are based on certain ideas known from the literature /1-11/ and culminate in a generalization of Routh's Theorem and its modification /1-6, 12, 13/.

1. Consider a dynamical system whose behaviour is described by ordinary differential equations of the following form:

$$\mathbf{x}^{\cdot} = \mathbf{f}(\mathbf{x}) \quad (\mathbf{x} \in \mathbf{R}^n, \ \mathbf{f}(\mathbf{x}) \in C^1\colon \mathbf{R}^n \to \mathbf{R}^n) \tag{1.1}$$

Assume that Eqs.(1.1) have first integrals which do not depend explicitly on time:

$$\mathbf{U}(\mathbf{x}) = \mathbf{c} \quad (\mathbf{c} \in \mathbf{R}^k, \ U(\mathbf{x}) \in C^2\colon \mathbf{R}^n \to \mathbf{R}^k) \tag{1.2}$$

*Prikl. Matem. Mekhan., 54, 6, 905-913, 1990

Given an arbitrary function $V(\mathbf{x}) \in C^2$: $\mathbf{R}^n \to \mathbf{R}^1$, we shall say that it takes a stationary value at constant values of the first integrals on a set $M \subset R^n$ and is not degenerate on that set, if M is the largest connected closed set on which the following equalities hold:

$$\delta V\,|_{\delta \mathbf{U}=0} = 0, \quad V = m = \text{const}, \quad \text{but} \quad \delta^2 V\,|_{\delta \mathbf{U}=0} \neq 0$$

Theorem 1.1. If a function $V(\mathbf{x})$ takes a stationary value at constant values of the first integrals (1.2) of system (1.1) on a set $M_0 \subset \mathbf{R}^n$ and is not degenerate on that set, and its total derivative with respect to time along trajectories of the system $V^{\cdot} = \langle \operatorname{grad} V, \mathbf{f} \rangle$, takes a stationary value on a set $N_0 \subset \mathbf{R}^n$ and $M_0 \subseteq N_0$, then M_0 is an invariant set of the system.

Proof. Suppose that the (stationary) value of V on the set M_0 is m_0, the corresponding constant values of the integrals (1.2) being $\mathbf{c}^0$. Then M_0 is determined by the relations

$$V_{,j} + \lambda_\beta U_{\beta,j} = 0, \; U_\alpha = c_\alpha{}^\circ, \quad V = m_0 \; (f_{,j} = \partial f / \partial x_j) \tag{1.3}$$

where $\lambda_1, \ldots, \lambda_k$ are undetermined Lagrange multipliers. In this section $i, j = 1, \ldots, n$; α, $\beta = 1, \ldots, k < n$, with the summation assumed throughout over repeated indices.

The set N_0 is determined by the relations

$$V_{,ij} f_j + V_{,j} f_{j i} = 0 \quad (V_{,ij} = \partial^2 V / \partial x_i \partial x_j) \tag{1.4}$$

and, moreover (by assumption), $M_0 \subseteq N_0$. Consequently (see (1.3) and (1.4)), the following conditions hold in M_0:

$$V_{,ij} f_j = -V_{,j} f_{j,i} = \lambda_\beta U_{\beta,j} f_{j,i} \tag{1.5}$$

Now, multiplying the left-hand side of the j-th equation in (1.3) by f_j, summing over j and using the fact that

$$U_{\alpha,j} f_j \equiv 0 \tag{1.6}$$

since $U_\alpha = c_\alpha$ are first integrals, we see that $V^{\cdot}$ vanishes throughout M_0.

Finally, differentiating the identity (1.6) with respect to x_i, we get

$$U_{\alpha,ij} f_j + U_{\alpha,j} f_{j,i} \equiv 0$$

whence it follows (see also (1.5)) that on M_0

$$\frac{d}{dt}(V_{,i} + \lambda_\beta U_{\beta,i}) \equiv V_{,ij} f_j + \lambda_\beta U_{\beta,ij} f_j = \lambda_\beta (U_{\beta,j} f_{j,i} + U_{\beta,ij} f_j) \equiv 0$$

Thus, all the Eqs.(1.3) that determine M_0 are invariant, and the function V is not degenerate on the set; in other words, M_0 is an invariant set.

Remark 1.1. Real solutions of system (1.1) in M_0 will be called stationary solutions, since they impart stationary values to the function V (at constant values of the first integrals) and to its total derivative with respect to time. In the general case such stationary solutions are time-dependent, but if $\dim M_0 = 0$ they are simply the singular points of system (1.1).

Remark 1.2. The invariant set M_0 over which V and its total time derivative take stationary values at constant values of the first integrals depends on these constants values; the stationary solutions $\mathbf{x}^\circ(t) \in M_0(\mathbf{c})$ depend in addition on the initial conditions $\mathbf{x}^\circ \in M_0(\mathbf{c})$. Hence the stationary solutions $\mathbf{x}^\circ(t, \mathbf{c}, \mathbf{x}^\circ)$ form a family of dimension not less than the sum of the number of arbitrary constants among $\mathbf{c}$ which are independent for $M_0(\mathbf{c})$ and the number of arbitrary initial conditions among $\mathbf{x}^\circ \in M_0(\mathbf{c})$ which are independent for $\mathbf{x}^\circ(t, \mathbf{c}, \mathbf{x}^\circ)$; the function $V^{\cdot}$ vanishes, of course, on this family.

Remark 1.3. The function V (for fixed values of $\mathbf{c}$) and its total time derivative may take stationary values not only on the sets M_0 and N_0, respectively, but also, in general, on sets M_1 and N_1 $(M_1 \subseteq N_1)$, M_2 and N_2 $(M_2 \subseteq N_2), \ldots$. To the sets $M_1, M_2, \ldots$ there correspond (by assumption) the same values of the constants $\mathbf{c}$, but generally different values of V (obviously, $V^{\cdot}$ vanishes, as before, on all these sets). The sets $M_1, M_2, \ldots$ also depend on the constants $\mathbf{c}$, while the stationary solutions $\mathbf{x}^k(t, \mathbf{c}, x^k) \subset M_k$ $(k = 1, 2, \ldots)$ form families of appropriate dimensions.

2. The stability of the invariant sets of system (1.1) (stationary solutions) described in Sect.1 can be investigated using Lyapunov's direct method, by means of the following theorems, which are analogues of Routh's Theorem /1-6/ and its modification /12, 13/:

Theorem 2.1. If a function $V(\mathbf{x})$ has a local strict minimum (maximum) at constant values $\mathbf{c}^\circ$ of the first integrals of the system on a compact set $M_0(\mathbf{c}^\circ)$, and its total time derivative $V^{\cdot}$ along the trajectories of the system has a local maximum (minimum) on the same set, then $M_0(\mathbf{c}^\circ)$ is a stable invariant set (and in that case any stationary solution $\mathbf{x}^\circ(t) \subset M_0(\mathbf{c}^\circ)$ is stable with respect to $\operatorname{dist}(\mathbf{x}, M_0(\mathbf{c}^\circ))$).

Theorem 2.2. If a function $V(\mathbf{x})$ takes a local strict minimum (maximum) at constant values $\mathbf{c}^\circ$ of the first integrals on a compact set $M_0(\mathbf{c}^\circ)$, and its total time derivative $V^\cdot$ along the trajectories of the system has a local maximum (minimum) on the family of sets $M_0(\mathbf{c})$ for all $\mathbf{c}$ sufficiently close to $\mathbf{c}^\circ$, then $M_0(\mathbf{c}^\circ)$ is a stable invariant set, and every solution sufficiently close to the invariant set $M_0(\mathbf{c}^\circ)$ tends asymptotically as $t \to \infty$ to the set $M_0(\mathbf{c})$ for the perturbed values of the first integrals (and in that case any solution $\mathbf{x}^\circ(t) \subset M_0(\mathbf{c}^\circ)$ is asymptotically stable with respect to $\operatorname{dist}(\mathbf{x}, M_0(\mathbf{c}))$); if, additionally, the constant integrals are not perturbed, then $M_0(\mathbf{c}^\circ)$ is an asymptotically stable invariant set (and in that case any solution $\mathbf{x}^\circ(t) \subset M_0(\mathbf{c}^\circ)$ is asymptotically stable with respect to $\operatorname{dist}(x, M_0(\mathbf{c}^\circ))$).

Theorem 2.3. If a function $V(\mathbf{x})$ does not have even a non-strict minimum (maximum) at constant values $\mathbf{c}^\circ$ of the first integrals of the system on a compact invaraint set $M_0(\mathbf{c}^\circ)$, but its total time derivative along trajectories of the system has a local strict maximum (minimum) on the family of sets $M_0(\mathbf{c})$ for all $\mathbf{c}$ sufficiently close to $\mathbf{c}^\circ$, then $M_0(\mathbf{c}^\circ)$ is an unstable invariant set.

Remarks. 2.1. It is not assumed in the statements of Theorems 2.1 and 2.2 that V is non-degenerate on M_0, because if V and its total derivative $V^\cdot$ have extrema (with opposite signs) on M_0, the invariance of the latter can be proved without this condition (see below).

2.2. The invariance of the set $M_0(\mathbf{c}^\circ)$ referred to in Theorem 2.3 is understood in the sense that $M_0(\mathbf{c}^\circ)$ satisfies the conditions of Theorem 1.1.

We will first prove Theorem 2.1. Let the set $M_0(\mathbf{c}^\circ)$ give $V(\mathbf{x})$ a minimum at constant values $\mathbf{c}^\circ$ of the first integrals (1.2): denote the minimum in question by $m_0(\mathbf{c}^\circ)$. Similarly, let $V^\cdot$ have a maximum - which is obviously zero (see Sect.1).

Consider an arbitrary solution $\mathbf{x}^\circ(t)$ of system (1.1) with initial data $\mathbf{x}^\circ(t_0) \subset M_0(\mathbf{c}^\circ)$ satisfying the conditions

$$\mathbf{U}(\mathbf{x}^\circ(t_0)) = \mathbf{c}^\circ \tag{2.1}$$

Obviously, $\mathbf{U}(\mathbf{x}^\circ(t)) \equiv \mathbf{c}^\circ$, since $\mathbf{U}(\mathbf{x}) = \mathbf{c}$ are first integrals. When that happens we have $V(\mathbf{x}^\circ(t)) \geqslant m_0(\mathbf{c}^\circ)$, since by assumption $m_0(\mathbf{c}^\circ)$ is a minimum of $V(\mathbf{x})$ at fixed values $\mathbf{c}^\circ$ of the constants $\mathbf{c}$ of the integrals (1.2).

On the other hand, $V^\cdot(x^\circ(t)) \leqslant 0$, since by assumption zero is a maximum of $V^\cdot(\mathbf{x})$, i.e.,

$$V(\mathbf{x}^\circ(t)) = V(\mathbf{x}^\circ(t_0)) + \int_{t_0}^{t} V^\cdot(\mathbf{x}^\circ(t))\,dt \leqslant m_0(\mathbf{c}^\circ)$$

Consequently, $V(\mathbf{x}^\circ(t)) \equiv m_0(\mathbf{c}^\circ)$ and $\mathbf{x}^\circ(t) \subset M_0(\mathbf{c}^\circ)$ $\forall t \geqslant t_0$, since $V(\mathbf{x})$ has the strict minimum $m_0(\mathbf{c}^\circ)$ on $M_0(\mathbf{c}^\circ)$.

Thus, $M_0(\mathbf{c}^\circ)$ is an invariant set (note that this is proved without appeal to the compactness of the set $M_0(\mathbf{c}^\circ)$).

To prove that the set $M_0(\mathbf{c}^\circ)$ is stable, we consider a set

$$\operatorname{dist}(\mathbf{x}, M_0(\mathbf{c}^\circ)) = \varepsilon \tag{2.2}$$

where $\varepsilon > 0$ is sufficiently small (but finite). Since the set $M_0(\mathbf{c}^\circ)$ is compact, the set (2.2) is also compact and the continuous function $V(\mathbf{x}) - m_0(\mathbf{c}^\circ)$ is always bounded below on this set by a negative number $-\sigma_1$ $(\sigma_1 > 0)$. Now, if the variables $\mathbf{x}$ satisfy the relations $\mathbf{U}(\mathbf{x}) = \mathbf{c}^\circ$, we have $V(\mathbf{x}) - m_0(\mathbf{c}^\circ) \geqslant \sigma_2 > 0$ on the same set, because $V(\mathbf{x})$ has a strict minimum $m_0(\mathbf{c}^\circ)$ on the set $M_0(\mathbf{c}^\circ)$ at constant values $\mathbf{c}^\circ$ of the integrals (1.2). By continuity, there exist positive number σ_3 and σ_4 such that, whenever $\|\mathbf{U} - \mathbf{c}^\circ\| < \sigma_3$, then $V - m_0(\mathbf{c}^\circ) > \sigma_4$. Then, choosing the positive number $\mu < \sigma_3/\sigma_1$, we see that the function

$$W = \mu(V - m_0(\mathbf{c}^\circ)) + \|\mathbf{U} - \mathbf{c}^\circ\|$$

is bounded below by a positive number $\sigma < \min(\sigma_3 - \sigma_1\mu, \mu\sigma_4)$ on the set (2.2). For this number σ, we can determine a number δ such that the domain

$$\operatorname{dist}(\mathbf{x}, M_0(\mathbf{c}^\circ)) < \delta \tag{2.3}$$

lies entirely in the interior of the domain $W < \sigma$, which in turn is contained in the interior of the domain

$$\operatorname{dist}(\mathbf{x}, M_0(\mathbf{c}^\circ)) < \varepsilon \tag{2.4}$$

Since W is a non-increasing function in this domain ($W^\cdot = \mu V^\cdot \leqslant 0$, because by assumption $V^\cdot$ has a maximum, equal to zero, on the set $M_0(\mathbf{c}^\circ)$), it follows that any perturbed solution of system (1.1) with initial data in the domain (2.3) will not leave the domain $W < \sigma$ or, a fortiori, the domain (2.4). Hence $M_0(\mathbf{c}^\circ)$ is a stable invariant set.

We will now prove Theorem 2.2. By Theorem 2.1, $M_0(\mathbf{c}^\circ)$ is a stable invariant set, such that any real solution $\mathbf{x}^\circ(t) \subset M_0(\mathbf{c}^\circ)$ is stable with respect to $\operatorname{dist}(\mathbf{x}, M_0(\mathbf{c}^\circ))$. Thus, any perturbed solution $\mathbf{x}(t)$ of system (1.1) lies entirely within the domain (2.4), however small the positive number ε, provided that the following inequality holds at the starting time t_0 (see (2.3)):

$$\| \mathbf{x}(t_0) - \mathbf{x}^\circ(t_0) \| < \delta = \delta(\varepsilon)$$

Since $V^\cdot(\mathbf{x})$ has a local strict maximum on the family of sets $M_0(\mathbf{c})$ for all $\mathbf{c}$ sufficiently close to $\mathbf{c}^\circ$, and this maximum is zero, it follows that $\varepsilon > 0$ can be chosen so small (but finite) that there will be no points in the domain (2.4) belonging to other invariant sets $M_1(\mathbf{c}), M_2(\mathbf{c}), \ldots,$ distinct from $M_0(\mathbf{c})$, if such sets exist (as already pointed out, $V^\cdot(\mathbf{x})$ vanishes on any such set).

Since $V(\mathbf{x}(t))$ is a non-increasing function in the domain (2.4), it must approach a limiting value v_0, never falling below that value:

$$V \geqslant v_0 \tag{2.5}$$

Suppose that the perturbed solution $\mathbf{x}(t)$ does not approach $M_0(\mathbf{c})$. Then there exists a sequence of points

$$\mathbf{x}^p = \mathbf{x}(t_0 + p\tau) \ (p = p_1, p_2, \ldots; \ p_1 < p_2 < \ldots; \ \tau = \text{const} > 0) \tag{2.6}$$

such that $\operatorname{dist}(\mathbf{x}^p, M_0(\mathbf{c})) \geqslant \gamma > 0$, where γ is some number, possibly small, but finite. Considering the sequence (2.6) in the bounded domain (2.4), we can extract a subsequence

$$\mathbf{x}^s = \mathbf{x}(t_0 + s\tau) \quad (s = p_{s_1}, p_{s_2}, \ldots; \ s_1 < s_2 < \ldots) \tag{2.7}$$

which converges to some point $\mathbf{x}^*$, such that (by continuity)

$$V(\mathbf{x}^*) = v_0, \quad \operatorname{dist}(\mathbf{x}^*, M_0(\mathbf{c})) \geqslant \gamma$$

Now consider solutions $\mathbf{x}^*(t)$ and $\mathbf{x}^s(t)$ emanating at the starting time from points $\mathbf{x}^*$ and $\mathbf{x}^s$, respectively. Since $\operatorname{dist}(\mathbf{x}^*, M_0(c)) \geqslant \gamma$ and $V^\cdot \equiv 0$ (in the domain (2.4)) only when $\mathbf{x} \in M_0(\mathbf{c})$, there must exist a time $t_1 > t_0$ such that

$$V(\mathbf{x}^*(t_1)) = v_1 < v_0 \tag{2.8}$$

Next, since the sequence $\mathbf{x}^s$ converges to $\mathbf{x}^*$, it follows (since the solutions depend continuously on the initial data) that

$$\| \mathbf{x}^*(t_1) - \mathbf{x}^s(t_1) \| < \alpha, \quad \forall s > s_*(\alpha)$$

for any prescribed number $\alpha > 0$.

Then (by continuity)

$$| V(\mathbf{x}^s, t_1)) - V(\mathbf{x}^*(t_1)) | < \beta, \quad \forall s > s_*(\alpha) \equiv s_*(\alpha(\beta)) \equiv s^*(\beta)$$

for any prescribed number $\beta > 0$.

Choosing $\beta > v_0 - v_1$, we obtain the inequality

$$V(\mathbf{x}^s(t_1)) < v_1 + \beta < v_0$$

which may be rewritten in the form

$$V(\mathbf{x}(t_1 + s\tau)) < v_0, \quad \forall s > s^* \tag{2.9}$$

because the right-hand sides of system (1.1) are independent of time and therefore $\mathbf{x}^s(t_1) \equiv \mathbf{x}(t_1 + s\tau)$.

Obviously, inequality (2.9) contradicts (2.5), i.e., our assumption must be false. Hence any perturbed solution $\mathbf{x}(t)$ sufficiently close to the invariant set $M_0(\mathbf{c}^\circ)$ (see (2.4)) tends as $t \to \infty$ to a set $M_0(\mathbf{c})$ corresponding to perturbed constant values of the first integrals (1.2): $\mathbf{c} = \mathbf{U}(\mathbf{x}(t_0))$.

If the constants of the integrals (1.2) are not perturbed, we conclude, repeating the procedure outlined above, that the perturbed solution tends to the set $M_0(\mathbf{c}^\circ)$ as $t \to \infty$.

Finally, let us prove Theorem 2.3, assuming that the function $V(\mathbf{x})$ does not have even a non-strict minimum on $M_0(\mathbf{c}^\circ)$, but $V^\cdot(\mathbf{x})$ has a local strict maximum on the family of sets $M_0(\mathbf{c})$. The function $V(\mathbf{x}) - m_0(\mathbf{c}^\circ)$ may then become negative in the neighbourhood of $M_0(\mathbf{c}^\circ)$.

Consider a perturbed solution $\mathbf{x}(t)$ with initial data satisfying (2.1), such that

$$V(\mathbf{x}(t_0)) < m_0(\mathbf{c}^\circ), \quad \| \mathbf{x}(t_0) - \mathbf{x}^\circ(t_0) \| < \delta$$

where $\delta > 0$ is arbitrarily small $(\mathbf{x}^\circ(t_0) \in M_0)$. Under these conditions, of course,

$$\operatorname{dist}(\mathbf{x}(t_0), M_0(\mathbf{c}^\circ)) > 0 \tag{2.10}$$

since otherwise $\mathbf{x}(t_0) \in M_0(\mathbf{c}^\circ)$ and $V(\mathbf{x}(t_0)) = m_0(\mathbf{c}^\circ)$.

Suppose that $\mathbf{x}(t)$ never leaves the domain (2.4), where $\varepsilon > 0$ is some sufficiently small (but finite) number. As before, choose ε so small that the domain (2.4) contains no points of other invariant sets $M_1(\mathbf{c}), M_2(\mathbf{c}), \ldots$, if such exist. Then $V^{\cdot}(\mathbf{x}(t)) < 0$.

The function $V(\mathbf{x}(t))$ is bounded in the domain (2.4) and, since it is a decreasing function, it must have a limit, remaining constantly not less than the limit (see inequality (2.5)). Since $\mathbf{x}(t) \not\subset M_0(\mathbf{c}^\circ)$ (see (2.10)), there exists a sequence (2.6) such that $\operatorname{dist}(\mathbf{x}^p, M_0(\mathbf{c}^\circ)) \geqslant \gamma > 0$, where γ it some possibly small but finite number; in the bounded domain (2.4) we can extract from this sequence a subsequence (2.7) which converges to some point $\mathbf{x}^*$, where, by continuity,

$$\operatorname{dist}(\mathbf{x}^*, M_0(\mathbf{c}^\circ)) \geqslant \gamma \quad V(\mathbf{x}^*) = v_0$$

Now consider solutions $\mathbf{x}^*(t)$ and $\mathbf{x}^s(t)$ emanating at the starting time from points $\mathbf{x}^*$ and $\mathbf{x}^s$, respectively. Since $\mathbf{x}^* \notin M_0(\mathbf{c}^\circ)$, there is a time $t_1 > t_0$ at which inequality (2.8) is satisfied. Repeating the procedure outlined in the proof of Theorem 2.2, we obtain (2.9), contradicting (2.5). This means that our assumption was false, i.e., the invariant set $M_0(\mathbf{c}^\circ)$ is unstable.

Under these conditions, clearly, the following is a sufficient condition for any solution $\mathbf{x}^\circ(t) \subset M_0(\mathbf{c}^\circ)$ to be unstable: there exists a time t_* such that an arbitrarily small neighbourhood of the point $\mathbf{x}^\circ(t_*) \in M_0(\mathbf{c}^\circ)$, contains points $\mathbf{x} \in R^n$ for which $V(\mathbf{x}) - m_0(\mathbf{c}^\circ) < 0$ (in particular, if $V(\mathbf{x})$ has a maximum (minimum) on the set $M_0(\mathbf{c}^\circ)$ (at constant $\mathbf{c}^\circ$), then, all other assumptions of Theorem 2.3 being valid, all solutions $\mathbf{x}^\circ(t) \subset M_0(\mathbf{c}^\circ)$ are unstable).

3. The results presented above can obviously be extended to systems of type (1.1) with no first integrals. In rigorous terms, we have the following theorems.

Theorem 3.1. If a function $V(\mathbf{x}) \in C^2\colon \mathbf{R}^n \to \mathbf{R}^1$ takes a stationary value on some set $M_0 \subset R^n$, where $\operatorname{rank}(\partial^2 V/\partial \mathbf{x}^2) = n - \dim M_0$, and the total time derivative of the function along trajectories of system (1.1), $V^{\cdot} = \langle \operatorname{grad} V, \mathbf{f} \rangle$, takes a stationary value on a set N_0 such that $M_0 \subseteq N_0$, then M_0 is an invariant set of the system.

Theorem 3.2. If a function $V(\mathbf{x})$ has a local strict minimum (maximum) on a compact set M_0, and its total time derivative along trajectories of system (1.1) has a /strict/ local maximum (minimum) on the same set, then M_0 is an /asymptotically/ stable invariant set (and in that case any stationary solution $\mathbf{x}^\circ(t) \subset M_0$ is /astmptotically/ stable with respect to $\operatorname{dist}(\mathbf{x}, M_0)$).

Theorem 3.3. If some function $V(\mathbf{x})$ does not have even a non-strict minimum (maximum) on a compact invariant set M_0, but its total time derivative along trajectories of system (1.1) has a local strict maximum (minimum) on M_0, then M_0 is an unstable invariant set.

Remarks. 3.1. Theorem 3.2 states, in particular, that the main stability theorem of Lyapunov's direct method furnishes conditions not only for the stability of given solutions, but also for the existence of stable solutions of dynamical systems; an analogous remark holds with regard to Theorems 2.1 and 2.2 for systems with known first integrals.

3.2. If V and $V^{\cdot}$ take uniformly extremal values on M_0 (of appropriate signs), the assumption in Theorems 3.2 and 3.3 that this set is compact may be dropped.

Example. Consider the motion of a rigid body about its centre of mass. Let $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ be the principal central tensor of inertia and $\mathbf{M} = (M_1, M_2, M_3)$ the angular momentum vector. All vector and tensor quantities are specified in terms of their projections on the principal axes of inertia. The equations of motion of the body about its centre of mass, driven by forces with a torque $\mathbf{Q}$, are

$$M^{\cdot} = M \times \partial H/\partial M + Q \;(H = {}^1\!/_2\,(I_1^{-1}M_1^2 + I_2^{-1}M_2^2 + I_3^{-1}M_3^2)) \tag{3.1}$$

Let $\mathbf{Q} = (M_1 q, p, M_2 q)$, where $q(\mathbf{M})$ and $p(\mathbf{M})$ are smooth functions, $I_1 < I_2 < I_3$, $F = (I_1^{-1} - I_2^{-1})M_1^2 + (I_3^{-1} - I_2^{-1})M_3^2$.

The function $V = {}^1\!/_2 F^2$ defines an invariant set $M_0 \equiv \{F = 0\}$ which is not a manifold. If $q \leqslant 0$ in a neighbourhood of the invariant set M_0, then it is stable in Lyapunov's sense; if $q < 0$ or $q > 0$, and moreover $\lim q(\mathbf{M}) \neq 0$ as $F \to 0$, then M_0 is asymptotically stable or unstable, respectively.

4. Let $F_i(\mathbf{x})$ $(i = 0, \ldots, l)$ be first integrals as in (1.2), where the functions F_i are assumed to be homogeneous in the generalized sense:

$$(\partial F_i/\partial x)\cdot Kx \equiv \chi_i F_i, \quad \forall x \in R^n, \quad K = \operatorname{diag}(k_1, \ldots, k_n) \tag{4.1}$$

Consider the system of equations

$$\mathbf{x}^{\cdot} = \mathbf{f}(\mathbf{x}) + \mathbf{K}\mathbf{x}\varphi(\mathbf{x}) \tag{4.2}$$

where $\varphi\colon R^n \to R$ is an arbitrary continuous function.

Proposition 4.1. The domains $\{F_i > 0\}$, $\{F_i < 0\}$ and surfaces $\{F_i = 0\}$ are invaraint under the action of the flow (4.2).

The system of Eqs.(4.2) has l general, time-independent first integrals. If $\varphi(\mathbf{x}) \equiv \varphi(F_0, \ldots, F_l)$, the system also has a conditional first integral which depends explicitly on time.

Proof. Differentiate the functions F_i along trajectories of system (4.2). The functions F_i are first integrals of Eqs.(1.1) and satisfy (4.1). Then

$$dF_i/dt = (\partial F_i/\partial x)\cdot(\mathbf{f}(\mathbf{x}) + \mathbf{K}\mathbf{x}\varphi(\mathbf{x})) \equiv \chi_i F_i \varphi(\mathbf{x}) \tag{4.3}$$

Consequently,

$$F_i(\mathbf{x}(t)) = F_i(\mathbf{x}(0)) \exp\left[\int_0^t \varphi(x(\tau))\, d\tau\right]$$

and the functions F_i maintain their signs on any solution of Eqs.(4.2). Hence the domains $\{F_i > 0\}$, $\{F_i < 0\}$ and surfaces $\{F_i = 0\}$ are invariant under the flow (4.2). This proves the first part of the proposition.

Let us investigate the function

$$J_{pq} = |F_p|^{\chi_q}/|F_q|^{\chi_p}, \quad p, q = 0, \ldots, l \tag{4.4}$$

on the set $G_{pq} = \{F_p(\mathbf{x}) \neq 0\} \cap \{F_q(\mathbf{x}) \neq 0\}$. By (4.3), we see that along trajectories of system (4.2)

$$dJ_{pq}/dt = (\chi_q \operatorname{sign} F_p |F_p|^{\chi_q - 1} dF_p/dt\, |F_q|^{\chi_p} -$$
$$\chi_p \operatorname{sign} F_q |F_q|^{\chi_p - 1} dF_q/dt\, |F_p|^{\chi_q})\, |F_q|^{-2\chi_p} \equiv 0$$

and the functions J_{pq} are first integrals in the domains G_{pq}. The functions F_i are first integrals on the surfaces $\{F_i = 0\}$. Consequently, if $\mathbf{L} = \{0, \ldots, l\}$ and α is any subset of L, then on the set

$$\Gamma_{\alpha\beta} = \bigcap_{i\in\alpha} \{F_i = 0\} \cap \bigcap_{j\in\beta} \{F_j \neq 0\} \quad \beta = \mathbf{L} \setminus \alpha$$

the functions $F_i\ (i \in \alpha)$ and $J_{j\sigma}$, where σ is the least element of β, $j \in \beta \setminus \{\sigma\}$, form a system of l independent first integrals $I_1(\Gamma_{\alpha\beta}), \ldots, I_l(\Gamma_{\alpha\beta})$. This proves the second part of the proposition.

Let us now fix a joint level of the first integrals $I_p(\Gamma_{\alpha\beta})$

$$J = \{\mathbf{x}\colon F_i = 0,\ i \in \alpha;\ J_{j\sigma} = q_j,\ j \in \beta \setminus \{\sigma\}\} \tag{4.5}$$

Then, solving (4.5) for $j \in \beta \setminus \{\sigma\}$ as equations in F_j, we have $F_j = V_j(F_q, q_j)$. Consequently, on $\Gamma_{\alpha\beta}$,

$$\varphi(F_0, \ldots, F_l) \equiv \Phi(F_\sigma, q_j), \quad j \in \beta \setminus \{\sigma\}$$
$$dF_\sigma/dt = \chi_\sigma F_\sigma \Phi(F_\sigma, q_j) \tag{4.6}$$

The function

$$J_\sigma(F_\sigma(\mathbf{x}), t, q_j) = 0 \tag{4.7}$$

which is a general integral of (4.6), defines a conditional first integral of Eqs.(4.2) which depends explicitly on time, which it was required to prove.

Proposition 4.2. Assume that the function $\mathbf{f}(\mathbf{x})$ on the right of Eqs.(1.1) satisfies the condition

$$f_i(\delta_1 x_1, \ldots, \delta_n x_n) = \delta_1^{\gamma_{i1}} \cdot \ldots \cdot \delta_n^{\gamma_{in}} f_i(x_1, \ldots, x_n), \quad \forall \mathbf{x} \in R^n$$

Then, if

$$\sum_{j=1}^{n} k_j \gamma_{ij} - k_i \equiv \alpha = \text{const}, \quad \forall i \in \{1, \ldots, n\}$$

Eq.(4.2) may be reduced, by a suitable transformation of the space and time variables, to the form (1.1) in the domain $R^n \setminus \{\mathbf{x}\colon F_i = 0,\ i = 0, \ldots, l\}$.

Proof. Let F_i be any of the first integrals of Eqs.(1.1) which satisfy (4.1), say F_0. In the domain $\{F_0 > 0\}$ we seek a change of variables in the form

$$x_i = \Lambda_i F_0^{\alpha_i} y_i, \quad i = 1, \ldots, n \tag{4.8}$$

where Λ_i, α_i are unknown constants. By (4.3), for solutions of Eqs.(4.2),

$$x_i^{\cdot} = \Lambda_i \alpha_i y_i \chi_0 F_0^{\alpha_i} \varphi + \Lambda_i y_i^{\cdot} F_0^{\alpha_i} \tag{4.9}$$

Substitute (4.8) and (4.9) into (4.2). By our assumption about the right-hand side of Eqs.(1.1),

$$\begin{gathered} \Lambda_i \alpha_i y_i \chi_0 F_0^{\alpha_i} \varphi + \Lambda_i y_i^{\cdot} F_0^{\alpha_i} = \\ (\Lambda_1 F_0^{\alpha_1})^{\gamma_{i1}} \ldots (\Lambda_n F_0^{\alpha_n})^{\gamma_{in}} f_i(y_1, \ldots, y_n) + k_i \Lambda_i y_i F_0^{\alpha_i} \varphi \end{gathered} \tag{4.10}$$

Then, if $\alpha_i = k_i \chi_0^{-1}$, $\Lambda_i = \exp \alpha_i$, it follows from the assumptions that Eqs.(4.10) may be expressed in the form

$$y_i^{\cdot} = \beta f_i(y_1, \ldots, y_n), \quad \beta = \exp(\alpha / \chi_0) F_0^{\alpha/\chi_0} \tag{4.11}$$

Applying a time transformation $t \to \tau$ such that

$$d\tau = \beta dt$$

we reduce Eqs.(4.11) to the form (1.1). Reasoning in the same way for the domain $\{F_0 < 0\}$, and then also for the functions $F_1, \ldots, F_l$, we obtain a collection of transformations applying throughout $R^n \setminus \{\mathbf{x}: F_i = 0, i = 0, \ldots, l\}$, as required.

Corollary. If $\alpha = 0$, system (4.2) can be reduced to the form of (1.1) by one transformation of the space variables.

Example. We again consider the motion of a rigid body about its centre of mass. Suppose that the forces driving the body have zero torque. Then Eqs.(3.1) have integrals $F_0 = H$, $F_1 = \mathbf{M}^2$ and are completely integrable

Now let $\mathbf{Q} = \mathbf{M}\varphi(M_1, M_2, M_3)$, where $\varphi(\mathbf{M})$ is a continuous function. In this case Eqs. (3.1) have a time-independent first integral

$$J_1 = H/F_1$$

Given a fixed level of this integral, $\{J_1 = q_1\}$, if it is true that $\varphi(\mathbf{M}) = \Phi(F_0, F_1)$, we can also find a first integral which depends explicitly on time. This integral is obtained from the general solution of the equation

$$F_1^{\cdot} = 2F_1 \Phi(F_1 q_1, F_1)$$

Applying the transformation of space and time variables

$$\mathbf{M} = \mathbf{K}\sqrt{\mathbf{M}^2}, \quad d\tau = dt\sqrt{\mathbf{M}^2}$$

we obtain the Euler equations

$$d\mathbf{K}/d\tau = \mathbf{K} \times \mathbf{I}^{-1}\mathbf{K}$$

Eqs.(3.1) with torques of this form arise when one is investigating the motion of a body under the action of forces of resistance. Such equations have been studied for the case $\varphi \equiv \mathrm{const}$ (/2, Sect.147/).

The idea underlying our investigation of Eqs.(4.2) is similar to that employed by Elliot /14/, who considered a similar problem, concerning the reduction of the equations of motion of a point in a resistant medium to the canonical form of the Hamilton equations.

The authors are grateful to A.A. Kosov for useful discussions of the results reported in this paper.

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Translated by D.L.

PMM U.S.S.R.,Vol.54,No.6,pp.751-755,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

STEADY MOTIONS AND INTEGRAL MANIFOLDS OF SYSTEMS WITH QUADRATIC INTEGRALS*

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An investigation is made of conservative systems with an additional integral of motion which is quadratic in the velocity. A method which takes into account the specific features of the mechanical problems is proposed to describe steady motions and integral surfaces in phase space. As an example, a non-holonomic problem, involving the motion of a rigid body carrying a gyroscope is considered.

Topological analysis of mechanical systems with known integrals $F_1,\ldots,F_k$ aims at describing the surfaces in phase space defined by fixed values of the integrals and studying the bifurcations of these surfaces /1/. The bifurcation points are defined by a dependence condition involving the integrals, $\Sigma\lambda_i dF_i=0$ (λ_i (where λ_i are Lagrange multipliers), or $dF_\lambda=0$, where $F_\lambda=\Sigma\lambda_iF_i$ is a pencil of integrals with constant coefficients λ_i. The condition $dF_\lambda=0$ is invariant /2/, i.e., it holds along the whole trajectory of the system emanating from a critical point of the pencil F_λ. The motion in this case is said to be steady. Such motions have been studied by numerous authors, e.g., /3-7/. In the typical case they form families parametrized by the values of the constants λ_i.

Thus, topological analysis involves the description of steady motions. When the integrals (other than the entry) are linear in the velocity, both problems can be tackled by means of reduced potentials /1, 8/. In this paper, consideration will be given to functions which play an analogous role for a conservative system with an additional integral which is a quadratic function of the velocity.

1. Let M be a configurational manifold with Riemannian form $\langle\cdot,\cdot\rangle$. In order to include the non-holonomic case, our phase space will be an m-dimensional subbundle $T'M$ of the tangent bundle TM: at every point $x\in M$ the fibre $T_x'M$ of this subbundle is the space of velocities allowable by the constraints (in the holonomic case $T'M=TM$). Assume that the integrals are

$$H(\mathbf{v})={}^1/_2\langle\mathbf{v},\mathbf{v}\rangle+V(x),\ F(\mathbf{v})={}^1/_2\langle\Gamma\mathbf{v},\mathbf{v}\rangle+\langle\mathbf{a},\mathbf{v}\rangle+W(x)$$

where $\mathbf{v}\in T'M$ is the velocity vector at the point $x\in M$, V and W are functions of the positional variables, Γ is a symmetric linear bundle operator, and $\mathbf{a}$ is a vector field on M. We may assume that Γ acts from $T'M$ to $T'M$ and that $\mathbf{a}\in T'M$; otherwise we replace them respectively by $Pr\circ\Gamma$ and $Pr(\mathbf{a})$, where Pr is the bundle operator of orthogonal projection

*Prikl.Matem.Mekhan.,54,6,914-919,1990